Highsted Grammar School

A – Level

Enrichment Booklet

"We all use maths every day. To predict weather…to tell time…to handle money. Maths is more than formulas and equations. It's logic; it's rationality. It's using your mind to solve the biggest mysteries we know."

Introduction to Mathematics A-Level Enrichment at

Highsted Grammar School.

Dear Students,

This booklet has been designed to take you beyond the A-Level Maths curriculum and enhance your problem solving skills. In it, you will find a variety of problem solving questions, which are broken down into different skills so that you are able to pick areas which you have a keen interest in. The idea behind it is that you can think about the Maths that you have learnt in school in a different context. These problems come from a variety of resources including the Advanced Maths Support Program and the UK Maths Challenge. At the end of this booklet, there is also a recommended reading list for you to have a look at to further extend your Mathematical knowledge.

I hope that you enjoy this and if you want to discuss any of the problems with your teachers then please do contact them.

Kind Regards,

Mrs L Allen

Assistant Headteacher

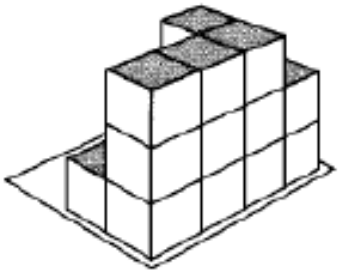
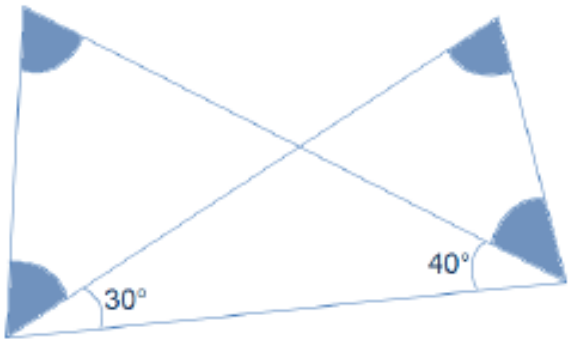
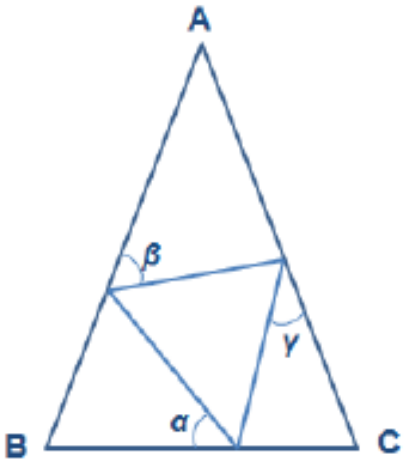
Leader of Mathematics

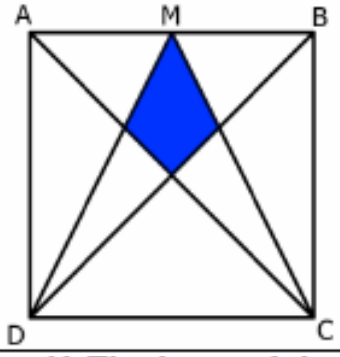
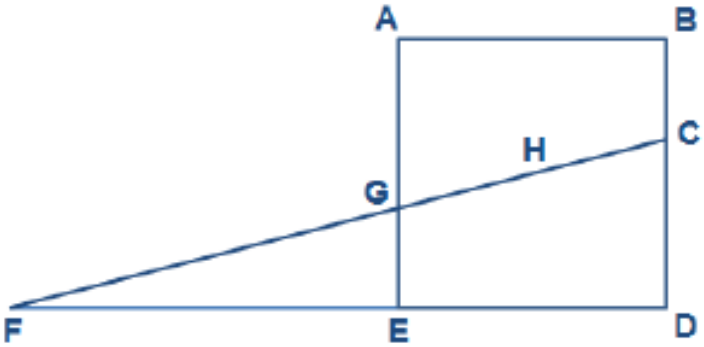
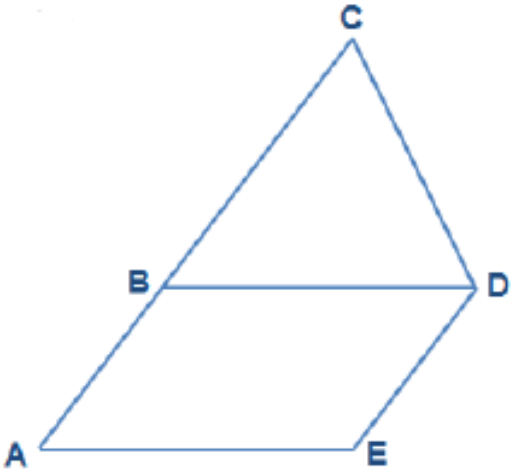
CONTENTS

Number Problems

1	What number, when multiplied by itself, is equal to 27 x 147?	
2	The average of three numbers is 8. Two of the numbers are 5 and 13. What is the other number?	
3	5! means 5 x 4 x 3 x 2 x 1. In general n! means n x (n – 1) x (n – 2) x…x 2 x 1. What is 101! divided by 99! ?	
4	What is the smallest number divisible by 1, 2, 3, 4, 5, 6, 7, 8 and 9?	
5	What is the value of the following expression? $$\frac{(101^4-4)(101^4-1)}{(101^2-2)(101^2-1)}-\frac{(101^4-4)(101^4-1)}{(101^2-2)(101^2+1)}$$ *This problem was set by the United Kingdom Mathematics Trust for the Senior Team Mathematics Challenge. Other questions from the challenge are available http://www.furthermaths.org.uk/stmchallengepast.php*	www.ukmt.org.uk/
6	Simplify the following expression giving your answer as a number $$\frac{(10!+9!)(8!+7!)(6!+5!)(4!+3!)(2!+1!)}{(10!-9!)(8!-7!)(6!-5!)(4!-3!)(2!-1!)}$$ *This problem was set by the United Kingdom Mathematics Trust for the Senior Team Mathematics Challenge. Other questions from the challenge are available http://www.furthermaths.org.uk/stmchallengepast.php*	www.ukmt.org.uk/

Shape and Angle Problems

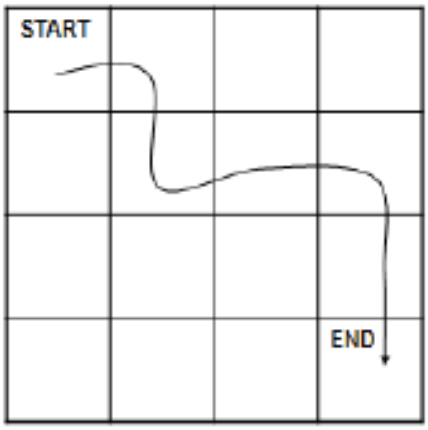
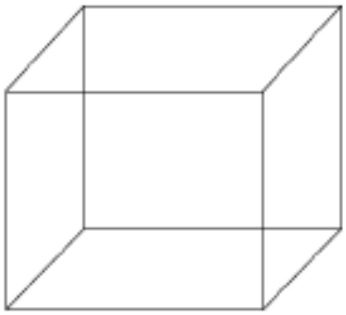
1	The solid shown rests on a flat surface. It is made from 1cm cubes placed, but not glued, together. Some of the cubes may be hidden. What is the minimum number of cubes required to make such a solid? 	www.ukmt.org.uk/
2	In the diagram below what is the sum of the four shaded angles? 	
3	In the diagram, triangle *ABC* is isosceles, with *AB* = *AC*. The inscribed triangle is equilateral. Find an expression for angle α in terms of angles β and γ. 	www.ukmt.org.uk/

4	*ABCD* is a square. *M* is the midpoint of the side *AB*. By constructing the lines *AC*, *MC*, *BD* and *MD*, the blue shaded quadrilateral is formed. What fraction of the total area is shaded?  	www.nrich.maths.org
5	*ABDE* is a square with centre *H*. The base of the square *DE* is extended so that it meets the straight line *CF* which passes through *H*.  If *BC* = 3cm and *CD* = 4cm find the area of the triangle *CDF*.	
6	In the diagram *ABDE* is a parallelogram and *ABC* is a straight line, *AB* = *x* cm, *BC* = *BD* = *y* cm.  If the area of the triangle *BCD* is *Q* cm^2, what is the area of the parallelogram?	www.ukmt.org.uk/

Quadratic & Cubic Problems

Question 1	Find the value of a such that the turning point of the parabola $y = x^2 - 2ax + 1$ is closest to the origin.	Source: UKMT
Question 2	Show that, if x is a real number then $2x^2 + 6x + 9$ is always positive.	
Question 3	Find the value of k such that the quadratic function of x $$k(x + 2)^2 - (x - 1)(x - 2)$$ is equal to zero for only one value of x.	
Question 4	Find all real solutions of the equation $(x^2 - 7x + 11)^{(x^2 - 11x + 30)} = 1$	Source: NRICH
Question 5	Find the smallest value of the function $f(x) = 2x^3 - 9x^2 + 12x + 3$ in the range $0 \le x \le 2$.	
Question 6	In this question b and c are real numbers. (a) By considering the graph of $y = x^2 + bx + c$ show that if $c < 0$ then the equation $x^2 + bx + c = 0$ has two distinct real roots. Is the converse statement also true?. (b) Determine the three conditions for the equation $x^2 + bx + c = 0$ to have distinct positive roots.	

Counting Problems

1	If you move only to the square immediately to the right or immediately below the one you are in, how many different routes are there from the square marked START to the square marked END in this diagram? 	
2	What is the probability that if you choose two socks at random from a drawer containing two red socks and two green socks, you get a pair of socks of the same colour?	
3	There are 43 girls and 37 boys in the sixth form. 45 of them are in year 12 and 35 of them are in year 13. How many more girls are there in year 12 then there are boys in year 13?	
4	Eight points are placed at the vertices of a cube. In total, how many triangles can be formed by joining three of these points?  *This problem was set by the United Kingdom Mathematics Trust for the Senior Team Mathematics Challenge. Other s from the challenge are available http://www.furthermaths.org.uk/stmchallengepast.php*	www.ukmt.org.uk/

Sequence & Series Problems

1	Show that, for any natural number n, a) $n(n+1)$ is even b) $n^3 - n$ is a multiple of 6 c) $n(n+1)(2n+1)$ is a multiple of 6	
2	By writing $n^3 + 11n$ as $n(n^2 - 1) + 12n$ show that every term of the sequence $n^3 + 11n$ is divisible by 6. Show that every term of the sequence $n^3 + 5n + 18$ is divisible by 6.	www.nrich.maths.org
3	In a sequence of numbers the nth term is given by $n^2 + 2n$. (a) Write the first six numbers in the sequence; (b) Explain why the numbers in the sequence alternate between odd and even. (c) How many numbers in the sequence are prime? Explain your reasoning	
4	⋮ ⋮ ⋮ ⋮ ⋮ ⋮ 22 30 39 49 ⋮ ⋮ 16 23 31 40 50 ⋮ … … 11 17 24 32 41 51 … … 7 12 18 25 33 42 52 … 4 8 13 19 26 34 43 … 2 5 9 14 20 27 35 … 1 3 6 10 15 21 28 … The natural numbers are entered in a grid in successive diagonals, as shown. The number 9 has grid position (3, 2). The number 16 has grid position (1, 6). What is the grid position of the number 1000?	www.ukmt.org.uk/

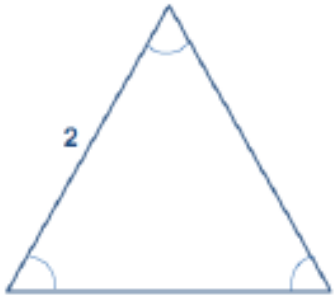
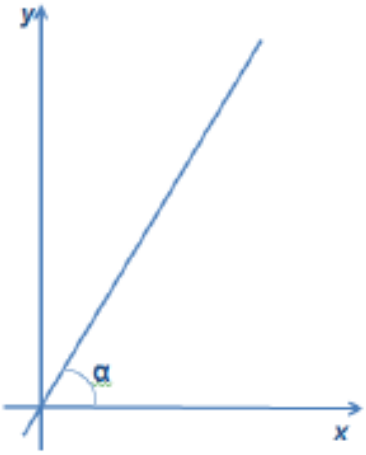
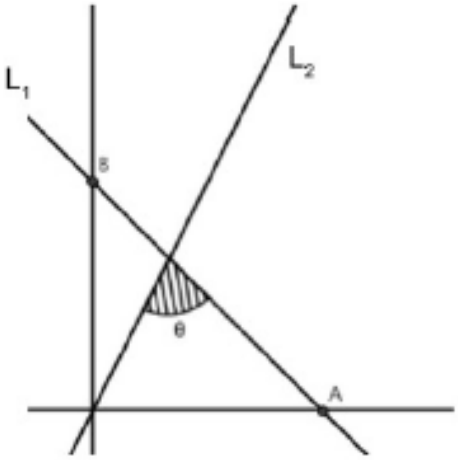
Algebra Problems

Question 1	Solve the system of equations: $ab = 1,\ bc = 2,\ cd = 3,\ de = 4,\ ea = 6$	Source: NRICH
Question 2	List all the real numbers x such that $(x - 5)^{x^2 - 4} = 1$	
Question 3	Five numbers are arranged in order from least to greatest: $x,\ x^3,\ x^4,\ x^2,\ x^0$ Where does $-x^{-1}$ belong in the list above?	
Question 4	Prove that are no positive integer solutions to the diophantine equation $x^2 - y^2 = 1$.	
Question 5	Suppose that a, b and c are integers satisfying the equation $a^3 + 3b^3 = 9c^3$. Explain why a must be divisible by 3. Show further that b and c must also be divisible by 3. Deduce that the only integer solution is $a = b = c = 0$.	Source: STEP
Question 6	Given the pair of simultaneous equations $ax + 2y = 1$ $2x + ay = b$ under what conditions on a, b does there exist (i) a unique solution, (ii) no solution, (iii) an infinite number of solutions?	

Coordinate Geometry Problems

Question 1	What is the area of the triangle whose vertices lie on the points (2,3), (-1, -2), (-9,10) ?	
Question 2	What is the equation of the set of points that are equidistant from (2, 4) and (1, 1)?	
Question 3	A and C are the opposite vertices of a square ABCD, and have coordinates (a,b) and (c,d), respectively. In terms of a, b, c and d, what are the coordinates of the other two vertices?	Source: NRICH
Question 4	Show that for every integer k the point $\left(\frac{2k}{k^2+1}, \frac{k^2-1}{k^2+1}\right)$ is on the circle with radius 1 and whose centre is the origin. Hence deduce that there are infinitely many points on the unit circle with rational coordinates.	
Question 5	Find the equation of the circle which passes through the points (1, 4), (7, 5) and (1, 8).	
Question 6	Find the equation of the circle which touches the line 3y – 4x – 24 = 0 at the point (0, 8) and also passes through the point (7, 9).	

Trigonometry Problems

1	What is the area of the equilateral triangle shown? 	
2	The equation of the line shown in the diagram is $y = \sqrt{3}x$. What is the value of α? 	
3	All the vertices of a triangle with angles $\frac{\pi}{3}$ and $\frac{\pi}{6}$ are on a circle of radius $\frac{\sqrt{3}}{2}$. What is the length of the shortest side?	
4	The diagram shows two lines L_1: $y = -x + 1$ and L_2: $y = 2x$. Let θ be the acute angle between L_1 and L_2. Find $\tan(\theta)$. Hint – you may need to add a line to form a right-angled triangle which includes θ. 	

KS5 Reading List

Websites:

Plus Magazine – Online general interest articles relating to practical uses of mathematics https://plus.maths.org/content/

Nrich - a selection of rich tasks for developing subject content knowledge as well as mathematical thinking and problem-solving skills. https://nrich.maths.org/9088

Integral - resources to support A Level maths and further maths courses. https://integralmaths.org/ (please speak to your maths teacher for username and password).

STEP Preparation:

STEP support programme by the University of Cambridge to help students develop problem-solving skills and prepare for university entrance exams: https://maths.org/step/

STEP questions with solutions:
https://undergroundmathematics.org/browse?typeOption=ReviewQuestion&reviewQuestionType=6

Ebook: Advanced Problems in Mathematics by Stephen Siklos (**ISBN:** 9781783747764) - this book bridges the gap between school and university mathematics and prepares students for an undergraduate mathematics course. The questions analysed in this book are all based on past STEP questions and each question is followed by a comment and a full solution.
https://www.openbookpublishers.com/reader/1050#page/1/mode/2up

Books

How to Study for a Mathematics Degree by Lara Alcock (ISBN: 9780199661329) - this book explains what to expect at university

Mathematics: A Very Short Introduction by Timothy Gowers (ISBN: 9780192853615) - this book gives an idea of the scope and spirit of mathematics but is written in an accessible style.

Number: A Very Short Introduction by Peter M. Higgins (ISBN: 9780199584055) - this book unravels the world of numbers, demonstrating its richness

Seventeen Equations that Changed the World by Ian Stewart (ISBN: 9781846685316)

Concepts of Modern Mathematics by Ian Stewart (ISBN: 8601400596593)

Fermat's Last Theorem by Simon Singh (ISBN: 9781841157917)

Mathematics: From the Birth of Numbers by Jan Gullberg, Peter Hilton (ISBN: 9780393040029)

The Man Who Knew Infinity: Life of the Genius Ramanujan New Ed by Robert Kanigel (ISBN: 9780349104522)

A History of Mathematics, Third Edition 3rd Revised edition by Carl B. Boyer, Uta C. Merzbach (ISBN: 8601405590862)

The Math Book: From Pythagoras to the 57th Dimension, 250 Milestones in the History of Mathematics by Clifford A. Pickover (ISBN: 9781402788291)

The Mathematics of Ciphers: Number Theory and RSA Cryptography by S.C. Coutinho (ISBN: 9781568810829)

Things to Make and Do in the Fourth Dimension by Matt Parker (ISBN: 9780141975863)

Further reading lists

University of Cambridge recommended reading list https://www.maths.cam.ac.uk/documents/reading-list.pdf/

Nrich reading list: https://nrich.maths.org/9477